Random walks on partite complexes

Izhar Oppenheim

Ben-Gurion University

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Joint work with Zohar Grinbaum-Reizis

Terminology and notation

Denote X to be a pure n-dim. simplicial complex, connected + connected links. Define $C^k(X) = \{\phi : X(k) \rightarrow \mathbb{R}\}$, e.g., $C^0(X)$ are functions from vertices of X to $\mathbb{R}$.

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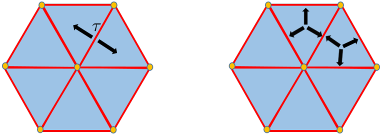
Define the following inner-product on $C^k(X)$:

$$\langle \phi, \psi \rangle = \sum_{\eta \in X(k)} w(\eta)\phi(\eta)\psi(\eta),$$

where w is a weight function which "takes into account" the higher dimensional structure (explicitly, $w(\tau) = (n-k)! \sum_{\sigma \in X(n), \tau \subseteq \sigma} w(\sigma), \ \forall \tau \in X(k)$).

Up and down operators

1. Up $k \nearrow l$ step $(l > k)$: For $\tau \in X(k)$ choose $\eta \in X(l)$ such that $\tau \subseteq \eta$ at random (according to the weight function w). Denote $U_{k \nearrow l} : C^k(X) \to C^l(X)$.
2. Down $l \searrow k$ step $(l > k)$: For $\eta \in X(l)$ choose at random $\tau' \in X(k)$ such that $\tau' \subseteq \eta$. Denote $D_{l \nearrow k} : C^l(X) \to C^k(X)$.



Random walks on simplicial complexes

Operators of the form $D_{l \nearrow k} U_{k \nearrow l} : C^k(X) \rightarrow C^k(X)$ or $U_{k \nearrow l} D_{l \nearrow k} : C^l(X) \rightarrow C^l(X)$ are averaging operators corresponding to random walks (with the same non-trivial spectrum).

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Most studied:

- $D_{n \nearrow 0} U_{0 \nearrow n} =$ the $\frac{1}{n+1}$-lazy random walk on the 1-skeleton (For non-lazy, take $\frac{n+1}{n}(D_{n \nearrow 0} U_{0 \nearrow n} - \frac{1}{n+1} I)$).
- $M_k = D_{k+1 \nearrow k} U_{k \nearrow k+1}$

Local spectral expanders

A graph is called

1. (One sided) λ-expander - connected + the spectrum of the rw is in $[-1, \lambda] \cup \{1\}$
2. Two sided λ-expander - connected + the spectrum of the rw is in $[-\lambda, \lambda] \cup \{1\}$

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A simplicial complex is called:

1. (One sided) λ-local spectral expander - all the links are connected + the spectrum of the rw in every links is in $[-1, \lambda] \cup \{1\}$
2. Two sided λ-local spectral expander - all the links are connected + the spectrum of the rw in every links is in $[-\lambda, \lambda] \cup \{1\}$

Random walks on simplicial complexes - local to global results

- Trickling down Theorem ([O. 2018]): Local spectral expansion can be deduced from expansion of 1 dim. links. Recently generalized by [Abdolazimi-Liu-Gharan].

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  1. Local spectral expansion $\Rightarrow$ a bound in the on second e.v. of M_k
  2. Two sided local spectral expansion $\Rightarrow$ e.v. $(\neq 0)$ of M_k are concentrated in strips around $\frac{k+1-j}{k+2}, j = -1, ..., k$.

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- More advanced machinery - See Liu's and Anari's talks.

Decomposition for two sided local spectral expanders - [Kaufman-O. 2018]

The weighting implies that $U_{j-1 \nearrow k} = U_{j \nearrow k} U_{j-1 \nearrow j}$. Thus $\operatorname{Im}(U_{j-1 \nearrow k}) \subseteq \operatorname{Im}(U_{j \nearrow k})$ and

$$\operatorname{Im}(U_{k-1 \nearrow k})^{\perp} \oplus \left(\operatorname{Im}(U_{k-1 \nearrow k}) \cap \operatorname{Im}(U_{k-2 \nearrow k})^{\perp}\right) \oplus \left(\operatorname{Im}(U_{k-2 \nearrow k}) \cap \operatorname{Im}(U_{k-3 \nearrow k})^{\perp}\right) \oplus \ldots$$

is an orthogonal decomposition of $C^k(X)$.

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For X that is a two-sided λ-local spectral expander (λ small), the following holds: For every $\phi \in \operatorname{Im}(U_{j \nearrow k}) \cap \operatorname{Im}(U_{j-1 \nearrow k})^{\perp}$, $\| M_k \phi - \frac{k+1-j}{k+2} \phi \| \leq \varepsilon(j,k,\lambda) \| \phi \|$.

Problem with the assumption of two-sided spectral gap in the links

In many examples the links of dimension 1 are bipartite graphs:

- Ramanujan complexes
- Quotients of buildings
- Coset complexes (Kaufman-O., Friedgut-Iluz, O'Donnell-Pratt)

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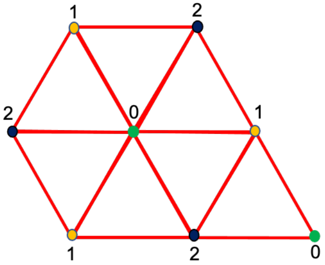
Can be circumvented considering $\sqrt{n}$-skeletons (but this feels like cheating)

Partite simplicial complexes

An n-dim. simplicial complex X is called *partite* or *colorable* if its vertex set can be partitioned into $n+1$ sets $S_0, ..., S_n$ such that every $\sigma \in X(n)$ has a vertex in each of the sets $S_0, ..., S_n$.



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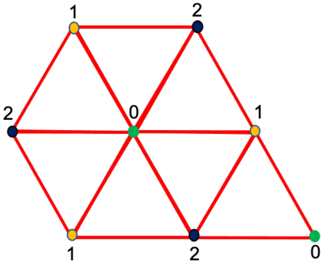

Define a type function on X: $\text{type}(\eta) = \{i : \eta \cap S_i \neq \emptyset\}$.

Partite decomposition (?)

For partite complexes, we have a finer decomposition of $C^n(X)$: For every $\nu \subsetneq \{0, ..., n\}$ define $(C^n(X))_\nu$ to be the subspace of $\phi \in C^n(X)$ such that

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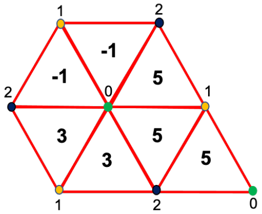
Example - $\phi \in (C^2(X))_{\{1\}}$:



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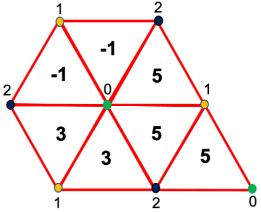

Note that $(C^n(X))_\nu$ is actually applying $U_{k \nearrow n}$ by type.

Partite decomposition (2) (?)

We note that if $\nu' \subseteq \nu$, then $(C^n(X))_{\nu'} \subseteq (C^n(X))_{\nu}$ and as in [K-O], we want to mod-out the parts of the space "coming from below".

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$$(C^n(X))^{\nu} = (C^n(X))_{\nu} \cap \left(\bigcap_{\nu' \subsetneq \nu} ((C^n(X))_{\nu'})^{\perp} \right).$$

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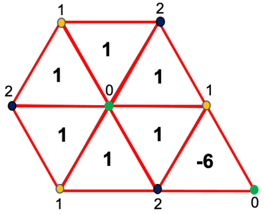
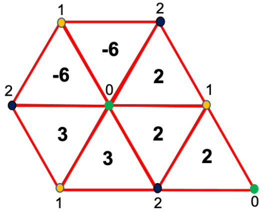
$$(C^n(X))^\nu = (C^n(X))_\nu \cap \left(\bigcap_{\nu' \subsetneq \nu} ((C^n(X))_{\nu'})^\perp \right).$$

However, note that two spaces $(C^n(X))^{\nu_1}, (C^n(X))^{\nu_2}$ need not be orthogonal if $\nu_1 \nsubseteq \nu_2$ or $\nu_2 \nsubseteq \nu_1$. Thus, we do not know if $C^n(X) = \bigoplus (C^n(X))^\nu$

Example: $\phi \in (C^2(X))^{\{0\}}$



$\psi \in (C^2(X))^{\{1\}}$

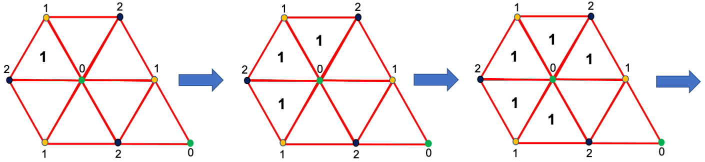


The spaces $C^n(X)_\nu$ as intersections

By our connectivity assumptions: For every $\nu \subsetneq \{0, ..., n\}, |\nu| < n$ it holds that

$$C^n(X)_\nu = \bigcap_{\nu', \nu \subsetneq \nu' \subsetneq \{0,...,n\}} C^n(X)_{\nu'}$$

Example: $C^2(X)_{\{0\}} = C^2(X)_{\{0,1\}} \cap C^2(X)_{\{0,2\}}$



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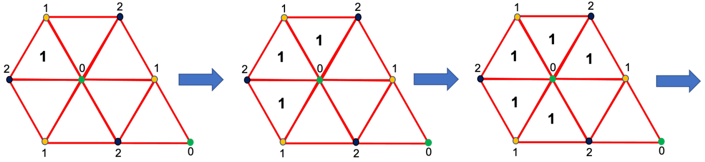

Actually,

$$C^n(X)_\nu = \bigcap_{\nu', \nu \subsetneq \nu' \subsetneq \{0,\ldots,n\}, |\nu'|=n} C^n(X)_{\nu'}$$

Restating the decomposition problem

Denote the subspaces

$$V_i = C^n(X)_{\{0,\ldots,n\} \setminus \{i\}}.$$

Thus every $C^n(X)_\nu$, with $\nu \subsetneq \{0, \ldots, n\}$ we have

$$C^n(X)_\nu = \bigcap_{i \notin \nu} V_i.$$

Restating the decomposition problem (2)

Let $\mathcal{H}$ be a Hilbert space and $V_0, ..., V_n$ be closed subspaces. Denote for every $\tau \subseteq \{0, ..., n\}$,

$$\mathcal{H}_\tau = \begin{cases} \bigcap_{i \notin \tau} V_i & \tau \neq \{0, ..., n\} \\ \mathcal{H} & \tau = \{0, ..., n\} \end{cases},$$

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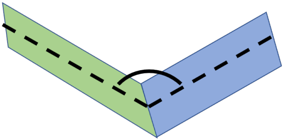
What criterion implies that $\mathcal{H} = \bigoplus_{\tau \subseteq \{0, \ldots, n\}} \mathcal{H}^\tau$?

Angles between subspaces

Definition

Let $V_1, V_2 \subseteq \mathcal{H}$ closed subspaces. Assume that $V_1 \not\subseteq V_2, V_2 \not\subseteq V_1$. Then the (cosine of) the angle between V_1, V_2 is defined as:

$$\cos(\angle(V_1, V_2)) = \sup\{|\langle x_1, x_2 \rangle| : x_i \in V_i \cap (V_1 \cap V_2)^\perp, \|x_i\| = 1\}.$$



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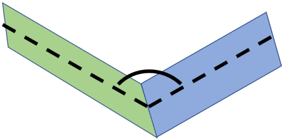

Fact: If P_V denotes the orthogonal projection on V, then

$$\cos(\angle(V_1, V_2)) = \|P_{V_1} P_{V_2} - P_{V_1 \cap V_2}\|.$$

Almost orthogonality implies decomposition

For subspaces $V_0, ..., V_n \subseteq \mathcal{H}$, if $\cos(\angle(V_i, V_j)) = 0$ for every $0 \leq i < j \leq n$, then the subspaces $\mathcal{H}^\tau, \tau \subseteq \{0, ..., n\}$ are all pairwise orthogonal and the decomposition is obvious.

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If the subspaces V_i are "almost orthogonal", we will still get a decomposition. In [DJ], the condition was that for every $0 \leq i < j \leq n$,

$$\cos(\angle(V_i, V_j)) < \frac{13}{28^n}.$$

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Remark: The [DJ] result actually talked about vanishing of cohomology (which is connected, but ignored in this talk) .

Decomposition through angles bound - two subspaces

For two subspace V_0, V_1, it is enough to have $\cos(\angle(V_0, V_1)) \leq \alpha < 1$ (or equivalently, $\angle(V_0, V_1) > 0$) to deduce a decomposition:

$$\mathcal{H}^{\emptyset} = V_0 \cap V_1, \mathcal{H}^{\{0,1\}} = (V_0 + V_1)^{\perp}, \mathcal{H}^{\{i\}} = V_{i+1} \cap (V_0 \cap V_1)^{\perp}.$$

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$$\begin{aligned} \|x^{\emptyset} + x^{\{0\}} + x^{\{1\}} + x^{\{0,1\}}\|^2 \geq \\ \|x^{\{0,1\}}\|^2 + \|x^{\emptyset}\|^2 + \|x^{\{0\}}\|^2 + \|x^{\{1\}}\|^2 - 2|\langle x^{\{0\}}, x^{\{1\}} \rangle| \geq \\ \|x^{\{0,1\}}\|^2 + \|x^{\emptyset}\|^2 + (2 - 2\alpha)(\|x^{\{0\}}\|^2 + \|x^{\{1\}}\|^2). \end{aligned}$$

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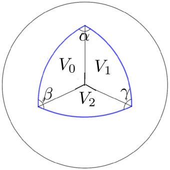
$$\|x^{\{0,1\}}\|^2 + \|x^{\emptyset}\|^2 + (2 - 2\alpha)(\|x^{\{0\}}\|^2 + \|x^{\{1\}}\|^2).$$

So $x^{\emptyset} + x^{\{0\}} + x^{\{1\}} + x^{\{0,1\}} = 0$ implies that all the summands are 0.

Decomposition through angles bound - intuition

Consider V_0, V_1, V_2 two-dimensional subspaces in $\mathbb{R}^3$ and consider the spherical triangle that arises from their intersection with the unit sphere:



One can think about $\angle(V_0, V_1, V_2)$ as the area of this triangle and then our guess for the criterion to the decomposition is $\angle(V_0, V_1, V_2) > 0$.

Decomposition through angles bound - intuition (2)

Fact: a triangle with angles α, β, γ is spherical iff the matrix

$$\begin{pmatrix} 1 & -\cos(\alpha) & -\cos(\beta) \\ -\cos(\alpha) & 1 & -\cos(\gamma) \\ -\cos(\beta) & -\cos(\gamma) & 1 \end{pmatrix}$$

is positive definite and its' determinant yields a bound on the spherical area of the triangle.

Decomposition through angles bound - intuition (3)

Dihedral angle in an n-simplex is an angle between two $n-1$ faces.

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Fact: An n-simplex with dihedral angles $\{\alpha_{i,j} : 0 \leq i, j \leq n\}$ is spherical iff the matrix

$$A_{i,j} = \begin{cases} 1 & i = j \\ -\cos(\alpha_{i,j}) & i \neq j \end{cases}$$

is positive definite and its' determinant yields a lower bound on the spherical volume of the simplex.

Decomposition Theorem

Theorem (Grinbaum-Reizis and Oppenheim 20')

Let $V_0, \ldots, V_n \subseteq \mathcal{H}$ be closed subspaces. If the matrix $A = A(V_0, \ldots, V_n)$ defined as

$$A_{i,j} = \begin{cases} 1 & i = j \\ -\cos(\angle(V_i, V_j)) & i \neq j \end{cases}$$

is positive definite, then $\mathcal{H} = \bigoplus_{\tau \subseteq \{0,\ldots,n\}} \mathcal{H}^{\tau}$. Moreover, the subspaces in the decomposition become "more orthogonal" as the smallest eigenvalue of A approaches 1 (equivalently as the determinant approaches 1).

Remark: We heavily use ideas of Kassabov.

Decomposition Theorem - back to simplicial complexes

Recall that

$$V_i = C^n(X)_{\{0,\dots,n\} \setminus \{i\}}.$$

Denote

$$\lambda_{i,j} = \max_{\tau \in X(n-2), \text{type}(\tau) = \{0,\dots,n\} \setminus \{i,j\}} (\text{Second e.v. of rw on } X_\tau).$$

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Claim: $\cos(\angle(V_i, V_j)) = \lambda_{i,j}$.

Decomposition Theorem - back to simplicial complexes (2)

Define $A(X)$ to be the $(n+1) \times (n+1)$ matrix (indexed by $0, ..., n$) as

$$A(X)_{i,j} = \begin{cases} 1 & i = j \\ -\lambda_{i,j} & i \neq j \end{cases}.$$

By the decomposition Theorem: if $A(X)$ is positive definite, then

$$C^n(X) = \oplus_{\nu \subseteq \{0,...,n\}} (C^n(X))^\nu$$

and this decomposition become "more orthogonal" as the smallest eigenvalue of $A(X)$ approaches 1.

Example of $A(X)$ for X Ramanujan

Let X be a partite Ramanujan complexes of thickness $q+1$, then

$$A(X) = \begin{pmatrix} 1 & -\frac{\sqrt{q}}{q+1} & 0 & \cdots & 0 & 0 & -\frac{\sqrt{q}}{q+1} \\ -\frac{\sqrt{q}}{q+1} & 1 & -\frac{\sqrt{q}}{q+1} & 0 & 0 & \cdots & 0 \\ 0 & -\frac{\sqrt{q}}{q+1} & 1 & -\frac{\sqrt{q}}{q+1} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{\sqrt{q}}{q+1} & 0 & \cdots & 0 & 0 & -\frac{\sqrt{q}}{q+1} & 1 \end{pmatrix}$$

Using decomposition to analyse down-up walks

As in the non-partite case, when considering $U_{k \nearrow n} D_{n \searrow k}$ walks, the decomposition will lead to strips in the spectra of walks.

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1. Each $(C^n(X))^\nu$ gives a strip in the spectrum of the form $[\frac{n+1-|\nu|}{n+1} - \varepsilon_\nu, \frac{n+1-|\nu|}{n+1} + \varepsilon_\nu]$

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2. For each ν, ε_ν above can be bounded using the matrix $A(X)$ (using angle considerations again).

The $P_{C^n(X)_\nu}$ projection

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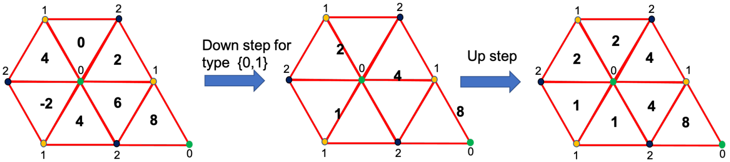

The $U_{n-1\nearrow n} D_{n\searrow n-1}$ walk

For $\phi \in C^n(X)$, the $U_{n-1\nearrow n} D_{n\searrow n-1}$ can be defined as following

$$U_{n-1\nearrow n} D_{n\searrow n-1}\phi = \frac{1}{n+1} \sum_{\nu \subsetneq \{0,\ldots,n\}, |\nu|=n} P_{C^n(X)_\nu}\phi,$$

where $P_{C^n(X)_\nu}$ is the orthogonal projection on $C^n(X)_\nu$.

Bound on spectra of $U_{n-1 \nearrow n} D_{n \searrow n-1}$ in terms of angles

Note that for $\phi \in C^n(X)^\tau$ for $\tau \subsetneq \{0, ..., n\}$, the following holds for every ν with $|\nu| = n$:

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Note that for $\phi \in C^n(X)^\tau$ for $\tau \subsetneq \{0, ..., n\}$, the following holds for every ν with $|\nu| = n$:

- If $\tau \subseteq \nu$, then $P_{C^n(X)_\nu} \phi = \phi$ (recall $C^n(X)_\tau \subseteq C^n(X)_\nu$).

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- If $\tau \subseteq \nu$, then $P_{C^n(X)_\nu} \phi = \phi$ (recall $C^n(X)_\tau \subseteq C^n(X)_\nu$).
- If $\tau \not\subseteq \nu$, then $\| P_{C^n(X)_\nu} \phi \| \leq \cos(\angle(C^n(X)_\nu, C^n(X)_\tau)) \| \phi \|$ (Recall $C^n(X)^\tau \subseteq (C^n(X)_\tau \cap (C^n(X)_{\nu \cap \tau})^\perp))$.

Bound on spectra of $U_{n-1\nearrow n} D_{n \searrow n-1}$ in terms of angles

Note that for $\phi \in C^n(X)^\tau$ for $\tau \subsetneq \{0, ..., n\}$, the following holds for every ν with $|\nu| = n$:

- If $\tau \subseteq \nu$, then $P_{C^n(X)_\nu} \phi = \phi$ (recall $C^n(X)_\tau \subseteq C^n(X)_\nu$).
- If $\tau \nsubseteq \nu$, then $\| P_{C^n(X)_\nu} \phi \| \leq \cos(\angle(C^n(X)_\nu, C^n(X)_\tau)) \|\phi\|$ (Recall $C^n(X)^\tau \subseteq (C^n(X)_\tau \cap (C^n(X)_{\nu \cap \tau})^\perp))$.

Thus for $\phi \in C^n(X)^\tau$,

$$\frac{1}{n+1} \sum_{\nu \subsetneq \{0,...,n\}, |\nu| = n} P_{C^n(X)_\nu} \phi =$$

$$\frac{n+1-|\tau|}{n+1} \phi + \frac{1}{n+1} \sum_{\nu, |\nu| = n, \tau \nsubseteq \nu} P_{C^n(X)_\nu} \phi$$

It follows that

$$\left\| \frac{1}{n+1} \sum_{\nu \subsetneq \{0,\dots,n\}, |\nu|=n} P_{C^n(X)_\nu} \phi - \frac{n+1-|\tau|}{n+1} \phi \right\|^2 \leq$$

$$\left(\frac{1}{n+1} \sum_{\nu, |\nu|=n, \tau \not\subseteq \nu} \cos(\angle(C^n(X)_\nu, C^n(X)_\tau)) \right)^2 \|\phi\|^2$$

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Recall that for ν with $|\nu| = n$, there is $C^n(X)_\nu = V_i$ and that $C^n(X)_\tau = \bigcap_{j \in \{0,\ldots,n\} \setminus \tau} V_j$.

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Question: How to bound $\cos(\angle(V_0, V_1 \cap ... \cap V_k))$ using $\cos(\angle(V_i, V_j))$?

$\sin(\angle(V_0, ..., V_k))$

Recall $A = A(V_0, ..., V_n)$ defined as

$$A_{i,j} = \begin{cases} 1 & i = j \\ -\cos(\angle(V_i, V_j)) & i \neq j \end{cases}.$$

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Recall $A = A(V_0, \ldots, V_n)$ defined as

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Denote

$$\sin^2(\angle(V_0, \ldots, V_k)) = \det(A(V_0, \ldots, A_k)).$$

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Denote

$$\sin^2(\angle(V_0, \ldots, V_k)) = \det(A(V_0, \ldots, A_k)).$$

Justification for the notation:

$$\sin^2(\angle(V_0, V_1)) = 1 - \cos^2(\angle(V_0, V_1)).$$

$\cos(\angle(V_0, V_1 \cap \ldots \cap V_k))$ bound Theorem

Theorem (Grinbaum-Reizis and Oppenheim 2021)

For closed subspaces $V_0, \ldots, V_k$ in a Hilbert space, it holds that

$$\sin^2(\angle(V_0, V_1 \cap \ldots \cap V_k)) \geq \frac{\sin^2(\angle(V_0, V_1, \ldots, V_k))}{\sin^2(\angle(V_1, \ldots, V_k))}.$$

In particular,

$$\cos(\angle(V_0, V_1 \cap \ldots \cap V_k)) \leq \sqrt{1 - \frac{\sin^2(\angle(V_0, V_1, \ldots, V_k))}{\sin^2(\angle(V_1, \ldots, V_k))}}.$$

Example in Ramanujan complexes

As noted before, for X that is a partite Ramanujan complex, we know the matrix $A(X)$.

$$A(X) = \begin{pmatrix} 1 & -\frac{\sqrt{q}}{q+1} & 0 & \cdots & 0 & 0 & -\frac{\sqrt{q}}{q+1} \\ -\frac{\sqrt{q}}{q+1} & 1 & -\frac{\sqrt{q}}{q+1} & 0 & 0 & \cdots & 0 \\ 0 & -\frac{\sqrt{q}}{q+1} & 1 & -\frac{\sqrt{q}}{q+1} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{\sqrt{q}}{q+1} & 0 & \cdots & 0 & 0 & -\frac{\sqrt{q}}{q+1} & 1 \end{pmatrix}$$

Example in Ramanujan complexes (2)

First example - the strip $[\frac{1}{n+1} - \varepsilon, \frac{1}{n+1} + \varepsilon]$ is achieved for $\phi \in C^n(X)^\tau$ with $|\tau| = n$. In that case

$$\varepsilon \leq \left(\frac{1}{n+1} \sum_{\nu, |\nu|=n, \tau \not\subseteq \nu} \cos(\angle(C^n(X)_\nu, C^n(X)_\tau)) \right) = \frac{2}{n+1} \frac{\sqrt{q}}{q+1}.$$

Example in Ramanujan complexes (3)

Second example - the strip $[\frac{n}{n+1} - \varepsilon, \frac{n}{n+1} + \varepsilon]$ is achieved for $\phi \in C^n(X)^{\{i\}}$. In that case

$$\varepsilon \leq \left(\frac{1}{n+1} \cos(C^n(X)_{\{0,\dots,n\} \setminus \{i\}}, C^n(X)_{\{i\}}) \right) \leq$$

$$\frac{2}{n+1} \frac{\sqrt{q}}{q+1} \left(\frac{Ax^{n-1}}{Ax^{n-1} + By^{n-1}} x + \frac{By^{n-1}}{Ax^{n-1} + By^{n-1}} y \right),$$

where A, B, x, y can be computed explicitly - all are positive and $x, y < 1$.

Example in Ramanujan complexes (3)

Second example - the strip $[\frac{n}{n+1} - \varepsilon, \frac{n}{n+1} + \varepsilon]$ is achieved for $\phi \in C^n(X)^{\{i\}}$. In that case

$$\varepsilon \leq \left(\frac{1}{n+1} \cos(C^n(X)_{\{0,\ldots,n\} \setminus \{i\}}, C^n(X)_{\{i\}}) \right) \leq \frac{2}{n+1} \frac{\sqrt{q}}{q+1} \left(\frac{Ax^{n-1}}{Ax^{n-1} + By^{n-1}} x + \frac{By^{n-1}}{Ax^{n-1} + By^{n-1}} y \right),$$

where A, B, x, y can be computed explicitly - all are positive and $x, y < 1$.
Similar bounds for the [Kaufman-O.] construction.

Concluding remarks

- In the case $n = 2$, our analysis of Ramanujan complexes is not tight by [Golubev-Parzanchevski]

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Concluding remarks

- In the case $n = 2$, our analysis of Ramanujan complexes is not tight by [Golubev-Parzanchevski]
- Other examples of complexes with sparse $A(X)$: quotients of buildings, O'Donnell-Pratt construction
- Work on other walks - ongoing

Thank you for listening