Fixed point spectrum of random groups

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The fixed point spectrum

Throughout: Γ is a finitely generated group.

- For a Banach space $\mathbb{E}$, Γ has $F\mathbb{E}$ if every affine isometric action of Γ on $\mathbb{E}$ has a fixed point.
- The (ℓ_p-) fix point spectrum of Γ is the set

$$\mathcal{F}_{\ell_\infty}(\Gamma) = \{p \in [1, \infty) : \Gamma \text{ has } F\ell_p\}$$

- (Czuron 14' ,Lavy & Olivier 14') $\mathcal{F}_{\ell_\infty}(\Gamma)$ is always in one of these forms:

$$\emptyset, [1, p_\Gamma], [1, p_\Gamma), [1, p_\Gamma] \setminus \{2\}, [1, p_\Gamma) \setminus \{2\}$$

The fixed point spectrum (2)

- If Γ has property (T), then $\mathcal{F}_{\ell_\infty}(\Gamma)$ is either $[1, p_\Gamma]$ or $[1, p_\Gamma)$ ($p_\Gamma \in (2, \infty]$).
- Several groups are known to have strong versions of property (T) that imply that their f.p. spectrum is $[1, \infty)$, e.g., by (Mimura 10') $\mathcal{F}_{\ell_\infty}(SL_n(\mathbb{Z}[x_1, ..., x_k])) = [1, \infty)$ for every $n \geq 4$.
- (Yu 05', Nica 13', Bourdon 16') If Γ is δ-hyperbolic, then there exists $p < \infty$ such that $p \notin \mathcal{F}_{\ell_\infty}(\Gamma)$. Bourdon: $p_\Gamma \leq$ the conformal dimension of $\partial_\infty \Gamma$.

Random groups in the triangular models

- For a fixed density $d \in (0,1)$, a random group in the triangular model $\mathcal{M}(m,d)$ is a group $\Gamma = \langle S | R \rangle$ with $|S| = m$ ($S \cap S^{-1} = \emptyset$) and R is a set of $\lfloor (2m-1)^{3d} \rfloor$ cyclically reduced relations of length 3 chosen randomly among all the sets with this cardinality.
- For a function $\rho(m)$, a random group in the **binomial** triangular model $\Gamma(m,\rho)$ is a group $\Gamma = \langle S | R \rangle$ with $|S| = m$ ($S \cap S^{-1} = \emptyset$) and R is a set relations of length 3 chosen independently with probability ρ.
- The model $\mathcal{M}(m,d)$ for a fixed $\frac{1}{3} < d < \frac{1}{2}$ "behaves the same as" $\Gamma(m,\rho)$ with $\rho = \frac{1}{(2m-1)^{3(1-d)}}$.

Properties of $\Gamma \in \Gamma(m,\rho)$

We say that $\Gamma \in \Gamma(m,\rho)$ has some group property P with over whelming probability (w.o.p) if

$$\lim_{m\to\infty} \mathbb{P}(\Gamma \in \Gamma(m,\rho) \text{ has P}) = 1$$

Fix $\frac{1}{3} < d < \frac{1}{2}$ and let $\rho = \frac{1}{(2m-1)^{3(1-d)}}$. The following holds for group $\Gamma \in \Gamma(m,\rho)$ w.o.p:

- (Ollivier) Γ is infinite and δ-hyperbolic.
- (Zuk) Γ has property (T).

Fixed point spectrum of $\Gamma \in \Gamma(m, \rho)$

From previous discussion, w.o.p there is $2 < p_\Gamma(d, m) < \infty$ such that

$$\mathcal{F}_{\ell_\infty}(\Gamma) = [1, p_\Gamma] \text{ or } [1, p_\Gamma)$$

Results regarding p_Γ:

- (Drutu & Mackay 17') There are constants c_d, C_d such that

$$c_d \sqrt{\frac{\log m}{\log \log m}} < p_\Gamma < C_d \log m$$

- (de Laat & de la Salle 18') Improved lower bound: $c_d \sqrt{\log m} < p_\Gamma$.
- (Oppenheim 21') Sharp lower bound: $c_d \log m < p_\Gamma$.

Two-sided spectral expanders (1)

- Let (V, E) be a finite graph and $\mathbb{E}$ be a Banach space.
- Define $\ell_2(V;\mathbb{E})$ to be the space of functions $\phi : V \rightarrow \mathbb{E}$ with norm

$$\|\phi\|^2 = \sum_{v \in V} \deg(v) \|\phi\|_{\mathbb{E}}^2$$

- Define $A_{\mathbb{E}}, M_{\mathbb{E}} : \ell_2(V;\mathbb{E}) \rightarrow \ell_2(V;\mathbb{E})$ by

$$A_{\mathbb{E}}\phi(v) = \frac{1}{\deg(v)} \sum_{u \sim v} \phi(u),$$

$$M_{\mathbb{E}}\phi \equiv \frac{1}{\sum_u \deg(u)} \sum_u \deg(u)\phi(u).$$

Two-sided spectral expanders (2)

For $\lambda \in \mathbb{R}$, we say that (V, E) is a $(\mathbb{E}, \lambda)$-two-sided spectral expander if

$$\| A_{\mathbb{E}} (I - M_{\mathbb{E}}) \|_{B(\ell_2(V;\mathbb{E}))} \leq \lambda$$

Remarks:

1. For every $\mathbb{E}$, every graph is a $(\mathbb{E}, 2)$-two-sided spectral expander.
2. For $\mathbb{E} = \mathbb{R}$ (or any Hilbert space), the definition coincides with the usual definition of what we'll call a classical λ-two-sided spectral expander: (V, E) is connected and the non-trivial spectrum of the SRW is contained in $[-\lambda, \lambda]$.

Riesz-Thorin Theorem and two-sided spectral expansion

(Riesz-Thorin) For every $p = \frac{2}{\theta}$, $0 < \theta < 1$, it holds for every graph (V, E) that

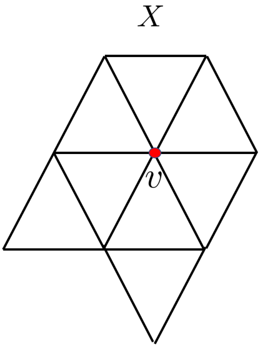
$$\|A_{\ell_p}(I - M_{\ell_p})\|_{B(\ell_2(V;\ell_p))} \leq \\ \|A_{\ell_2}(I - M_{\ell_2})\|^{\theta}_{B(\ell_2(V;\ell_2))} \|A_{\ell_\infty}(I - M_{\ell_\infty})\|^{1-\theta}_{B(\ell_2(V;\ell_\infty))} \leq \\ 2\|A_{\ell_2}(I - M_{\ell_2})\|^{\theta}_{B(\ell_2(V;\ell_2))}$$

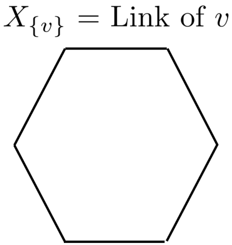
i.e., if (V, E) is a classical λ-two-sided expander, it follows that it is $(\ell_{\frac{2}{\theta}}, 2\lambda^{\theta})$-two-sided expander.

Link in a simplicial complex

For a simplex $v \in X(0)$, the **link of** v is a subcomplex

$$X_v = \{\eta \in X : v \notin \eta = \emptyset, \{v\} \cup \eta \in X\}.$$





Zuk's criterion for reflexive spaces

Theorem (Oppenheim 2020)

Let Γ f.g. group, X a simply connected 2-dim. complex, $\mathbb{E}$ a reflexive Banach space. Assume that $\Gamma \curvearrowright X$ geometrically. If there is $\lambda < \frac{1}{2}$ such that for every $v \in X(0)$, the link of v is a $(\mathbb{E}, \lambda)$-two-sided spectral expander, then Γ has $F\mathbb{E}$.

Remarks:

1. For $\Gamma \in \Gamma(m, \rho)$, the presentation complex X_Γ is a simply connected 2-dim. simplicial complex on which Γ acts geometrically.
2. For Hilbert spaces, this is weaker than the classical Zuk's criterion that requires only one-sided spectral gap.

Bound on p_Γ for $\Gamma \in \Gamma(m, \rho)$

Fix $\frac{1}{3} < d < \frac{1}{2}$ and $\rho = \frac{1}{(2m-1)^{3(1-d)}}$, and let $\Gamma \in \Gamma(m, \rho)$.

- (de Laat & de la Salle) w.o.p there is a constant L such that the link of a vertex in X_Γ is a classical $\frac{L}{m^{\frac{3}{2}d-\frac{1}{2}}}$-two-sided expander.
- Applying Riesz-Thorin, it follows that w.o.p for every $0 < \theta < 1$, the link of a vertex in X_Γ is a $(\ell_{\frac{2}{\theta}}, 2\frac{L^\theta}{m^{(\frac{3}{2}d-\frac{1}{2})\theta}})$-two-sided expander.
- Thus for every $p < \frac{\log(\frac{3}{2}d-\frac{1}{2})}{\log(4L)} \log(m)$, w.o.p there is $\lambda < \frac{1}{2}$ such that the link of a vertex in X_Γ is a (ℓ_p, λ)-two-sided expander.
- Applying the variation of Zuk's criterion above shows that w.o.p $p_\Gamma > c_d \log(m)$.

Proof - cohomological set up (1)

X 2-dim. simply connected, $\Gamma \curvearrowright X$ geom., $\mathbb{E}$ a Banach space and $\pi : \Gamma \rightarrow O(\mathbb{E})$ a representation. To avoid complications - also assume free action of Γ on X.

For $0 \leq k \leq 2$, the space of k-cochains twisted by π denoted by $C^k(X, \pi)$ is the space of all maps $\phi : \vec{X}(k) \rightarrow \mathbb{E}$ that are:

- Anti-symmetric: for every permutation $\sigma \in \text{Sym}(\{0, ..., k\})$ and every $(v_0, ..., v_k) \in \vec{X}(k)$,

$$\phi((v_{\sigma(0)}, ..., v_{\sigma(k)})) = (-1)^{\text{sgn}(\sigma)} \phi((v_0, ..., v_k)).$$

- Equivariant (w.r.t π): for every $g \in G$ and every $(v_0, ..., v_k) \in \vec{X}(k)$,

$$\phi(g.(v_0, ..., v_k)) = \pi(g) \phi((v_0, ..., v_k)).$$

Proof - cohomological set up (2)

Define the differential $d_k : C^k(X,\pi) \rightarrow C^{k+1}(X,\pi)$ by

$$d_k \phi((v_0, ..., v_{k+1})) = \sum_{i=0}^{k+1} (-1)^i \phi((v_0, ..., \hat{v}_i, ..., v_{k+1}))$$

Fact: $F\mathbb{E} \Leftrightarrow H^1(X,\pi) = \frac{\ker(d_1)}{\mathrm{Im}(d_0)} = 0$ for every π.

Proof - Norm and coupling

Define the norm on $C^k(X,\pi)$ as

$$\|\phi\|^2 = \sum_{\tau \in \Gamma \backslash \vec{X}(k)} m(\tau) |\phi(\tau)|_{\mathbb{E}}^2$$

where $m(\tau) = (2-k)! |\{\sigma \in X(2) : \tau \subseteq \sigma\}|$.
For the adjoint representation $\overline{\pi} : \Gamma \rightarrow O(\mathbb{E}^*)$, define $C^k(X,\overline{\pi})$ similarly and define $\overline{d_k} : C^k(X,\overline{\pi}) \rightarrow C^{k+1}(X,\overline{\pi})$. Define a coupling between $C^k(X,\pi)$ and $C^k(X,\overline{\pi})$ by

$$\langle \phi, \psi \rangle = \sum_{\tau \in \Gamma \backslash \vec{X}(k)} m(\tau) \langle \phi(\tau), \psi(\tau) \rangle$$

Proof - Nowak's Theorem

With the coupling above, take

$$d_k^* : C^{k+1}(X,\overline{\pi}) \to C^k(X,\overline{\pi}),$$
$$\overline{d_k}^* : C^{k+1}(X,\pi) \to C^k(X,\pi).$$

Theorem (Nowak 12')

Assume that $\mathbb{E}$ is reflexive and X, Γ as above. If there is a constant $C<1$ such that for every $\phi \in C^1(X,\pi), \psi \in C^1(X,\pi)$ it holds that

$$|\langle d_1\phi, \overline{d_1}\psi\rangle| + |\langle \overline{d_0}^*\phi, d_0^*\psi\rangle| \geq |\langle \phi, \psi\rangle| - C\frac{\|\phi\|^2+\|\psi\|^2}{2}$$

Then $H^1(X,\pi)=0$.

Proof - Garland's method

Localization for $\phi \in C^1(X,\pi)$ (or $\psi \in C^1(X,\overline{\pi})$)
$\phi_v(u) = \phi((v,u))$

$$2\|\phi\|^2 = \sum_{v \in \Gamma \backslash X(0)} \|\phi_v\|_v^2, 2\|\psi\|^2 = \sum_{v \in \Gamma \backslash X(0)} \|\psi_v\|_v^2$$

$$\langle \overline{d_0}^* \phi, d_0^* \psi \rangle = \sum_{v \in \Gamma \backslash X(0)} \langle (M_v)_{\mathbb{E}} \phi_v, \psi_v \rangle_v$$

$$\langle d_1 \phi, \overline{d_1} \psi \rangle = \langle \phi, \psi \rangle - \sum_{v \in \Gamma \backslash X(0)} \langle (A_v)_{\mathbb{E}} \phi_v, \psi_v \rangle_v .$$

Proof - computations

$$\langle d_1\phi, \overline{d_1}\psi\rangle + \langle \overline{d_0}^*\phi, d_0^*\psi\rangle =$$

$$\langle \phi, \psi\rangle - \sum_{v\in\Gamma\backslash X(0)} \langle (A_v)_{\mathbb{E}}\phi_v, \psi_v\rangle_v - \langle (M_v)_{\mathbb{E}}\phi_v, \psi_v\rangle_v =^{M_v = A_v M_v}$$

$$\langle \phi, \psi\rangle - \sum_{v\in\Gamma\backslash X(0)} \langle ((A_v)_{\mathbb{E}}(I - (M_v)_{\mathbb{E}})\phi_v, \psi_v\rangle_v$$

Proof - computations (2)

$$|\langle d_1\phi, \overline{d_1}\psi\rangle| + |\langle \overline{d_0}^*\phi, d_0^*\psi\rangle| \geq$$

$$|\langle \phi, \psi\rangle| - \sum_{v\in\Gamma\backslash X(0)} |\langle((A_v)_{\mathbb{E}}(I-(M_v)_{\mathbb{E}})\phi_v, \psi_v\rangle_v| \geq$$

$$|\langle \phi, \psi\rangle| - \sum_{v\in\Gamma\backslash X(0)} \|(A_v)_{\mathbb{E}}(I-(M_v)_{\mathbb{E}})\|\|\phi_v\|\|\psi_v\| \geq$$

$$|\langle \phi, \psi\rangle| - \sum_{v\in\Gamma\backslash X(0)} \lambda\frac{\|\phi_v\|^2+\|\psi_v\|^2}{2} =$$

$$|\langle \phi, \psi\rangle| - 2\lambda\frac{\|\phi\|^2+\|\psi\|^2}{2}$$

and we a done by Nowak's Theorem

Final remarks

- The same method applies for higher cohomology and other Banach spaces (commutative and non-commutative L^p spaces, uniformly curved spaces).
- The case where the links are bipartite is not treated in my method (was treated by Drutu and Mackay).

Thank you for listening