Vanishing of cohomology for groups acting on simplicial complexes

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Vanishing of cohomology and stability

Theorem (De Chiffre, Glebsky, Lubotzky, Thom)

Let Γ be a finitely presented group. If for every unitary representation π of Γ, $H^2(\Gamma, \pi) = 0$, then Γ is $(U(n), \|.\|_2)$-stable, when $\|.\|_2$ denotes Frobenius norm.

Main example of [DGLT]: Γ is (an extension of) a group acting geometrically on a Bruhat-Tits Building X.

Goal: We'll give machinery for proving vanishing of cohomology for groups acting on simplicial complexes.

Preliminary remark - Shapiro's Lemma

Below, we will usually not work with a discrete group Γ, but with G locally compact such that $\Gamma < G$ is a cocompact lattice. This is justified by Shapiro's Lemma:

Theorem (Shapiro's Lemma)

If G is a locally compact group and $\Gamma < G$ is a cocompact lattice, then for some k, if $H^k(G, \pi) = 0$ for every unitary representation π of G, then $H^k(\Gamma, \rho) = 0$ for every unitary representation ρ of Γ.

Simplicial complexes - Definition and notation

Abstract Definition: a simplcial complex X is a family of subsets of a set V such that if $\sigma \in X$ and $\tau \subseteq \sigma$, then $\tau \in X$.

We denote $X(k)$ to be subsets in X of size $k+1$. Throughout, X is n dimensional (i.e., $X(n+1) = \emptyset, X(n) \neq \emptyset$).

Geometrically: $X(0)$ - vertices, $X(1)$ - edges, $X(2)$ - triangles,...

We will denote by $\vec{X}(k)$ the ordered k simplices of X, e.g., if $\{v_1, v_2\} \in X(1)$, then $(v_1, v_2), (v_2, v_1) \in \vec{X}(1)$.

Group action

Let G be a topological group acting on X (by simplicial automorphisms).
We say that G acts on X geometrically if:

- The action is cocompact: X/G is finite.
- The action is proper: for every vertex $\{v\} \in X$, $G_{\{v\}} = \mathrm{Stab}(\{v\})$ is a compact subgroup of G.

In this lecture: group actions are always geometric.
Examples:

- $G = \{e\}$, X is finite.
- $G = \mathrm{SL}_{n+1}(\mathbb{Q}_p)$, X is the affine building arising from the BN-pair (we can also take G to be a cocompact lattice in $\mathrm{SL}_{n+1}(\mathbb{Q}_p)$).

Twisted cochains

Let G, X as above and $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of G ($\mathcal{H}$ is a Hilbert space, if G is topological group, π is always continuous).
For $0 \leq k \leq n$, the space of k-cochains twisted by π denoted by $C^k(X, \pi)$ is the space of all maps $\phi : \vec{X}(k) \rightarrow \mathcal{H}$ that are:

- Anti-symmetric: for every permutation $\sigma \in \text{Sym}(\{0, ..., k\})$ and every $(v_0, ..., v_k) \in \vec{X}(k)$,
$$\phi((v_{\sigma(0)}, ..., v_{\sigma(k)})) = (-1)^{\text{sgn}(\sigma)} \phi((v_0, ..., v_k)).$$
- Equivariant (w.r.t π): for every $g \in G$ and every $(v_0, ..., v_k) \in \vec{X}(k)$,
$$\phi(g.(v_0, ..., v_k)) = \pi(g)\phi((v_0, ..., v_k)).$$

Equivariant cohomology

For G, X, π as above, define the differential $d_k : C^k(X,\pi) \rightarrow C^{k+1}(X,\pi)$ by

$$d_k \phi((v_0, ..., v_{k+1})) = \sum_{i=0}^{k+1} (-1)^i \phi((v_0, ..., \hat{v}_i, ..., v_{k+1}))$$

(one should check that $d_k \phi \in C^{k+1}(X,\pi)$). Easy computation: $d_{k+1} d_k \equiv 0$. We define the equivariant cohomology as

$$H^k(X,\pi) = \frac{\ker(d_k)}{\operatorname{Im}(d_{k-1})}.$$

Fact: If X is contractible (and the action of G on X is geometric), then $H^k(X,\pi) = H^k(G,\pi)$.

Vanishing of cohomology - Motivation

Γ finitely presented, $H^2(\Gamma, \pi) = 0$, for every π implies stability w.r.t to Frobenius norm.

$H^1(G, \pi) = 0$ for every unitary representation π:

- Equivalent to Property (FH): Every affine isometric action of G on a Hilbert space has a fixed point.
- If G is locally compact, σ-compact: Equivalent to Property (T).

Garland's method: Deducing vanishing of cohomology from local properties of X.

Simplicial complexes - Terminology

X is:

- **Connected** if its 1-skeleton $X(0) \cup X(1)$ is connected (as a graph).
- **Pure *n*-dimensional** if $X(n+1) = \emptyset$ and every simplex of X is contained in an n-dimensional simplex.

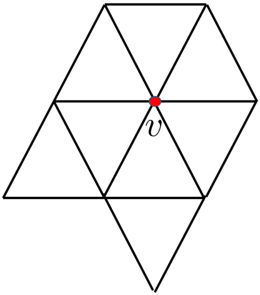
From now on: X is always connected, pure n-dimensional.

Simplicial complexes - links

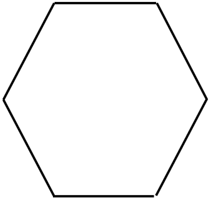
For a simplex $\tau \in X$, the **link of** τ is a subcomplex

$$X_\tau = \{\eta \in X : \tau \cap \eta = \emptyset, \tau \cup \eta \in X\}.$$

X



$X_{\{v\}} = \text{Link of } v$



Random walks on links

In this lecture: We will assume that "all" the links are connected and finite:

- For every $-1 \leq k < n-1$ and every $\tau \in X(k)$, X_τ is connected.
- For every $-1 < k \leq n-1$ and every $\tau \in X(k)$, X_τ is finite.

For every $\tau \in X(0) \cup ... \cup X(n-2)$, the 1-skeleton of X_τ is a connected finite graph.
Denote λ_τ to be the second e.v. of the random walk on the 1-skeleton of X_τ.
(I am cheating a little bit, since I consider weighted r.w. and not the simple r.w., whenever τ is of dimension $< n-2$).

Garland's method - general case

Theorem (Garland 73', Ballman-Swiatkowski 97')

Let X be a contractible, pure n-dimensional simplicial complex with connected and finite links and let G act geometrically on X. Fix $1 \leq k \leq n-1$. If

$$\max_{\tau \in X(k-1)} \lambda_\tau < \frac{1}{k+1},$$

then for every unitary representation π,
$H^k(X,\pi) = H^k(G,\pi) = 0$.

Garland's method - $n = 2$ case (Zuk's criterion)

Theorem (Zuk)

Let X be a pure 2-dimensional simplicial complex with connected and finite links and let G act geometrically on X. If for every vertex $\{v\} \in X(0)$, $\lambda_{\{v\}} < \frac{1}{2}$, then G has property (T).

Garland's method - general case (2)

Theorem (Oppenheim 12')

Let X and G as above. if

$$\max_{\tau \in X(n-2)} \lambda_\tau < \frac{1}{n},$$

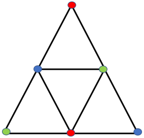
then for every unitary representation π *and every* $1 \leq k \leq n-1$, $H^k(X, \pi) = H^k(G, \pi) = 0$.

Trickling-down Theorem (Oppenheim 12'): Denote $\lambda_k = \max_{\tau \in X(k-1)} \lambda_\tau$, then

$$\lambda_k \leq \frac{\lambda_{n-1}}{1-(n-1-k)\lambda_{n-1}}.$$

Extra assumptions

A pure n-dimensional complex X is called $(n+1)$-partite / colorable if you can color the vertices with $n+1$ colors and each n-dimensional simplex has all the colors.



Assume now that X is $(n+1)$-partite and G acts on X preserving the coloring and X/G is a single n-dimensional simplex.

Example: G is a BN-pair group, X is the building ($G = SL_{n+1}(\mathbb{Q}_p)$, X is the affine $\tilde{A}_2$ building).

Subgroups and Subspaces

Abusing notation, fix $\triangle = \{0, ..., n\} \in X(n)$ (we denote the vertices by $0, 1, ..., n$). For every $\tau \subseteq \triangle$, denote
$G_\tau = \{g \in G : g.\tau = \tau\}$.
Note that $G_\emptyset = G$, and if $\tau' \subseteq \tau$, then $G_\tau \subseteq G_{\tau'}$.

Given a unitary representation π of G on a Hilbert space $\mathcal{H}$, define the following subspaces: for every $\tau \subseteq \triangle$,

$$\mathcal{H}_\tau = \mathcal{H}^{\pi(G_\tau)} = \{x \in \mathcal{H} : \forall g \in G_\tau, \pi(g).x = x\}.$$

Note that $\tau' \subseteq \tau$, then $\mathcal{H}_{\tau'} \subseteq \mathcal{H}_\tau$.

Subgroups and Subspaces (2)

Recall

$$\mathcal{H}_\tau = \mathcal{H}^{\pi(G_\tau)} = \{x \in \mathcal{H} : \forall g \in G_\tau, \pi(g).x = x\}.$$

Denote

$$\mathcal{H}^\tau = \mathcal{H}_\tau \cap \left(\bigcap_{\eta \subsetneq \tau} \mathcal{H}_\eta^\perp \right).$$

Decomposition implies vanishing of cohomology

Theorem (Dymara and Januszkiewicz, 02')

Let X, G be as above (X partite, $X/G \in X(n)$,...), and $\pi : G \to \mathcal{U}(\mathcal{H})$ a unitary representation. If for every $\tau \subseteq \triangle$,

$$\mathcal{H}_\tau = \bigoplus_{\eta \subseteq \tau} \mathcal{H}^\eta,$$

then $H^k(G, \pi) = 0$ for every $1 \leq k \leq n-1$.

Proof (2 dim. case) (1)

Assume that

$$\mathcal{H}_{\{0,1,2\}} = \bigoplus_{\tau \subseteq \{0,1,2\}} \mathcal{H}^{\tau}.$$

Let $\phi \in C^1(X, \pi) \cap \ker(d_1)$, we need to show there is $\psi \in C^0(X, \pi)$ such that $d_0 \psi = \phi$.

Observe: Since ϕ is equivariant and $\triangle$ is a fundamental domain, ϕ is determined by $\phi((0,1)), \phi((1,2)), \phi((2,0))$.

Proof (2 dim. case) (2)

Note: $d\phi = 0$ is equivalent to

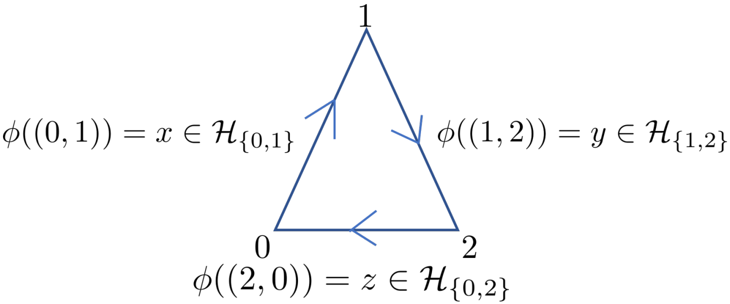
$$\phi((0,1)) + \phi((1,2)) + \phi((2,0)) = 0.$$

Also note, $\forall g \in G_{\{0,1\}}$,

$$\pi(g).\phi((0,1)) = \phi(g.(0,1)) = \phi((0,1)),$$

thus $\phi((0,1)) \in \mathcal{H}_{\{0,1\}}$.

Proof (2 dim. case) (3)



and $x + y + z = 0$.

Proof (2 dim. case) (4)

By the decomposition,

$$\mathcal{H}_{\{0,1\}} \ni x = x^{\{0,1\}} + x^{\{0\}} + x^{\{1\}} + x^{\emptyset},$$
$$\mathcal{H}_{\{1,2\}} \ni y = y^{\{1,2\}} + y^{\{1\}} + y^{\{2\}} + y^{\emptyset},$$
$$\mathcal{H}_{\{0,2\}} \ni z = z^{\{0,2\}} + z^{\{0\}} + z^{\{2\}} + z^{\emptyset},$$

where $x^\tau, y^\tau, z^\tau \in \mathcal{H}^\tau$. By the decomposition of $\mathcal{H}_{\{0,1,2\}}$, we get the following equations from $x + y + z = 0$:

$$x^{\{0,1\}} = y^{\{1,2\}} = z^{\{0,2\}} = 0,$$

$$x^{\{0\}} = -z^{\{0\}}, y^{\{1\}} = -x^{\{1\}}, y^{\{2\}} = -z^{\{2\}},$$

$$x^{\emptyset} + y^{\emptyset} + z^{\emptyset} = 0.$$

Proof (2 dim. case) (5)

Thus, we can define $\phi \in C^0(X, \pi)$ by

$$\psi(0) = x^{\{0\}} + x^{\emptyset} + y^{\emptyset},$$

$$\psi(1) = y^{\{1\}} + y^{\emptyset},$$

$$\psi(2) = z^{\{2\}}.$$

By the equations above, we get that $d_0 \psi = \phi$, e.g.,

$$d_0\psi((2,0)) = \psi(2) - \psi(0) = z^{\{2\}} - (x^{\{0\}} + x^{\emptyset} + y^{\emptyset}) =$$
$$z^{\{2\}} + z^{\{0\}} + z^{\emptyset} = \phi((2,0)).$$

Restating the problem

Recall: $\mathcal{H}_\tau = \mathcal{H}^{\pi(G_\tau)}$, $\tau \subseteq \{0,...,n\}$. Denote the subspaces

$$V_i = \mathcal{H}_{\{0,...,n\} \setminus \{i\}}.$$

Fact: Under our assumptions, for every $\tau \subsetneq \{0,...,n\}$,

$$G_\tau = \langle G_\sigma : \tau \subseteq \sigma, \sigma \subseteq \triangle, \sigma \in X(n-1) \rangle,$$

thus

$$\mathcal{H}_\tau = \bigcap_{\tau \subseteq \sigma, \sigma \in \triangle(n-1)} \mathcal{H}^{\pi(G_\sigma)} = \bigcap_{i \notin \tau} V_i.$$

Substituting, $\mathcal{H}$ with $\mathcal{H}_\triangle$, the decomposition can be rephrased as the following general question:

Restating the problem (2)

Let $\mathcal{H}$ be a Hilbert space and $V_0, ..., V_n$ be closed subspaces. Denote for every $\tau \subseteq \{0, ..., n\}$,

$$\mathcal{H}_\tau = \begin{cases} \bigcap_{i \notin \tau} V_i & \tau \neq \{0, ..., n\} \\ \mathcal{H} & \tau = \{0, ..., n\} \end{cases},$$

and

$$\mathcal{H}^\tau = \mathcal{H}_\tau \cap \bigcap_{\eta \subsetneq \tau} \mathcal{H}_\eta^\perp .$$

What criterion implies that $\mathcal{H} = \bigoplus_{\tau \subseteq \{0,...,n\}} \mathcal{H}^\tau$?

Angles between subspaces

Definition

Let $V_1, V_2 \subseteq \mathcal{H}$ closed subspaces. Assume that $V_1 \nsubseteq V_2, V_2 \nsubseteq V_1$. Then the (cosine of) the angle between V_1, V_2 is defined as:

$$\cos(\angle(V_1, V_2)) = \sup\{|\langle x_1, x_2 \rangle| : x_i \in V_i \cap (V_1 \cap V_2)^{\perp}, \|x_i\| = 1\}.$$

Fact: If P_V denotes the orthogonal projection on V, then

$$\cos(\angle(V_1, V_2)) = \|P_{V_1} P_{V_2} - P_{V_1 \cap V_2}\|.$$

Almost orthogonality implies decomposition

For subspaces $V_0, ..., V_n \subseteq \mathcal{H}$, if $\cos(\angle(V_i, V_j)) = 0$ for every $0 \leq i < j \leq n$, then the subspaces $\mathcal{H}^\tau, \tau \subseteq \{0, ..., n\}$ are all pairwise orthogonal and the decomposition is obvious.

Theorem (Dymara and Januszkiewicz, 02')

If the subspaces V_i are "almost orthogonal", we will still get a decomposition (and thus vanishing of cohomology). In [DJ], the condition was that for every $0 \leq i < j \leq n$,

$$\cos(\angle(V_i, V_j)) < \frac{13}{28^n}.$$

Remark: We do not state the whole strength of the [DJ] result.

Connection to the Garland's link condition

Proposition

Let G, X as above. Fix $\{0, ..., n\} \in X(n)$. Denote $\lambda_{i,j}$ to be the second largest e.v. of the random walk on the link of $\{0, ..., n\} \setminus \{i, j\}$.
Then for every unitary representation π, we have that

$$\cos(\angle(V_i(\pi), V_j(\pi))) \leq \lambda_{i,j}.$$

Thus, in [DJ], if

$$\max_{\tau \in X(n-2)} \lambda_\tau < \frac{13}{28^n},$$

then we get vanishing of cohomology for all unitary representations.

Decomposition through angles bound - two subspaces

For two subspace V_0, V_1, it is enough to have $\cos(\angle(V_0, V_1)) \leq \alpha < 1$ (or equivalently, $\angle(V_0, V_1) > 0$) to deduce a decomposition:

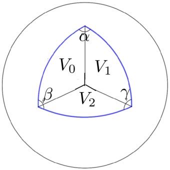
$$\mathcal{H}^{\emptyset} = V_0 \cap V_1, \mathcal{H}^{\{0,1\}} = (V_0 + V_1)^{\perp}, \mathcal{H}^{\{i\}} = V_{i+1} \cap (V_0 \cap V_1)^{\perp}.$$

$$\|x^{\emptyset} + x^{\{0\}} + x^{\{1\}} + x^{\{0,1\}}\|^2 \geq$$
$$\|x^{\{0,1\}}\|^2 + \|x^{\emptyset}\|^2 + \|x^{\{0\}}\|^2 + \|x^{\{1\}}\|^2 - 2|\langle x^{\{0\}}, x^{\{1\}} \rangle| \geq$$
$$\|x^{\{0,1\}}\|^2 + \|x^{\emptyset}\|^2 + (2 - 2\alpha)(\|x^{\{0\}}\|^2 + \|x^{\{1\}}\|^2).$$

So $x^{\emptyset} + x^{\{0\}} + x^{\{1\}} + x^{\{0,1\}} = 0$ implies that all the summands are 0.

Decomposition through angles bound - intuition

Consider V_0, V_1, V_2 two-dimensional subspaces in $\mathbb{R}^3$ and consider the spherical triangle that arises from their intersection with the unit sphere:



One can think about $\angle(V_0, V_1, V_2)$ as the area of this triangle and then our guess for the criterion to the decomposition is $\angle(V_0, V_1, V_2) > 0$.

Decomposition through angles bound - intuition (2)

Fact: a triangle with angles α, β, γ is spherical iff the matrix

$$\begin{pmatrix} 1 & -\cos(\alpha) & -\cos(\beta) \\ -\cos(\alpha) & 1 & -\cos(\gamma) \\ -\cos(\beta) & -\cos(\gamma) & 1 \end{pmatrix}$$

is positive definite and its' lowest e.v. yields a bound on the spherical area of the triangle.

Decomposition through angles bound - intuition (3)

Dihedral angle in an n-simplex is an angle between two $n-1$ faces.
Fact: An n-simplex with dihedral angles $\{\alpha_{i,j} : 0 \leq i, j \leq n\}$ is spherical iff the matrix

$$A_{i,j} = \begin{cases} 1 & i = j \\ -\cos(\alpha_{i,j}) & i \neq j \end{cases}$$

is positive definite and its' lowest e.v. yields a lower bound on the spherical volume of the simplex.

Decomposition Theorem

Theorem (Grinbaum-Reizis and Oppenheim 20')

Let $V_0, \dots, V_n \subseteq \mathcal{H}$ be closed subspaces. If the matrix

$$A_{i,j} = \begin{cases} 1 & i = j \\ -\cos(\angle(V_i, V_j)) & i \neq j \end{cases}$$

is positive definite, then $\mathcal{H} = \bigoplus_{\tau \subseteq \{0,\dots,n\}} \mathcal{H}^{\tau}$.

Remark: We heavily use ideas of Kassabov.

Vanishing of cohomology implication

Theorem (Grinbaum-Reizis and Oppenheim 20')

Let G, X as above. Fix $\{0, ..., n\} \in X(n)$. Denote $\lambda_{i,j}$ to be the second largest e.v. of the random walk on the link of $\{0, ..., n\} \setminus \{i, j\}$. If the matrix

$$A = \begin{pmatrix} 1 & -\lambda_{0,1} & -\lambda_{0,2} & \dots & -\lambda_{0,n} \\ -\lambda_{0,1} & 1 & -\lambda_{1,2} & \dots & -\lambda_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda_{0,n} & -\lambda_{1,n} & -\lambda_{2,n} & \dots & 1 \end{pmatrix}$$

is positive definite, then for every $1 \leq k \leq n-1$ and every unitary representation π, $H^k(G, \pi) = 0$.

Vanishing of cohomology for affine buildings

Corollary (Grinbaum-Reizis and Oppenheim 20')

Let G be a BN-pair group acting on an affine building X of dimension n. If X is non-thin (i.e., X is not a single apartment), then for every $1 \leq k \leq n-1$ and every unitary representation π, $H^k(G, \pi) = 0$.

Remark: This Corollary was already proven by Casselman by a different method.

How does it generalize to representations on Banach spaces (1)

Let G, X as above and π an isometric representation of G on some Banach space $\mathbb{E}$ (e.g., $\mathbb{E}$ is an L^p space or the space of $n \times n$ matrices with the p-Schatten norm).
What passes verbatim:

- The definition of equivariant cohomology.
- $H^k(X, \pi) = H^k(G, \pi)$.
- $\mathbb{E}_\tau = \{x \in \mathbb{E} : \forall g \in G_\tau, \pi(g).x = x\}$.
- Decomposition implies vanishing of cohomology (if we know how to define $\mathbb{E}^\tau$).

How does it generalize to representations on Banach spaces (2)

Problems:

- How to define $\mathbb{E}^\tau$? (Recall $\mathcal{H}^\tau = \mathcal{H}_\tau \cap (\bigcap_{\eta \subsetneq \tau} \mathcal{H}_\eta^\perp)$?
- What is the angle between subspaces now?
- Technical issues (not allowed to make arguments using passing to the orthogonal complement).

Solution: Pass to projections and not subspaces.

Let G be a locally compact unimodular group with a Haar measure μ.

For every function $f \in L^1(G)$ and every isometric representation π on a Banach space $\mathbb{E}$, we can always define $\pi(f) \in B(\mathbb{E})$ by

$$\pi(f).x = \int_G f(g)\pi(g).xdg.$$

Fix π. For any compact subgroup $K < G$, define P_K as $\pi(\frac{1_{G_\tau}}{\mu(G_\tau)})$. Note P_K is a projection on $E^{\pi(K)}$ and $\|P_K\| \leq 1$.

Projections and decomposition

Let G, X as above.

Fix $\triangle = \{0, ..., n\} \in X(n)$ and π an isometric representation on a Banach space $\mathbb{E}$.

For every $\tau \subseteq \triangle$, let $k_\tau \in L^1(G)$ defined as $k_\tau = \frac{1_{G_\tau}}{\mu(G_\tau)}$ (μ is the Haar measure of G). Denote that $P_\tau = \pi(k_\tau)$ and note that P_τ is a projection of norm 1 on

$$\mathbb{E}_\tau = \{x \in \mathbb{E} : \forall g \in G_\tau, \pi(g).x = x\}.$$

Projections and decomposition (2)

Define

$$\mathbb{E}^{\tau} = \mathrm{Im}(P_{\tau}) \cap (\bigcap_{\eta \subsetneq \tau} \ker(P_{\eta}).)$$

Thus in order to prove vanishing of cohomology, one should prove that

$$\mathbb{E}_{\tau} = \bigoplus_{\eta \subseteq \tau} \mathbb{E}^{\eta},$$

for every τ (when this is proven the proof of Dymara and Januszkiewicz for vanishing can be applied).

Angles between projections

Definition

Let $P_1, P_2 \in B(\mathbb{E})$ be two projections such that there is a projection $P_{1,2}$ with $\text{Im}(P_1) \cap \text{Im}(P_2) = \text{Im}(P_{1,2})$ and $P_{1,2}P_1 = P_{1,2}P_2 = P_{1,2}$. Define

$$\cos(\angle_{P_{1,2}}(P_1, P_2)) = \max\{\|P_1 P_2 - P_{1,2}\|, \|P_2 P_1 - P_{1,2}\|\}.$$

Vanishing of cohomology in Banach spaces

Theorem (Oppenheim 2017)

Let G, X as above and π be an isometric representation of G. Fix $\triangle \in X(n)$. There is a constant $\varepsilon = \varepsilon(n)$ such that if for every $\sigma, \sigma' \subseteq \triangle, \sigma, \sigma' \in X(n-1)$, it holds that $\cos(\angle(P_\sigma, P_{\sigma'})) \leq \varepsilon$, then

- *For every $\tau \subseteq \triangle$, $\mathbb{E}_\tau = \bigoplus_{\eta \subseteq \tau} \mathbb{E}^\eta$.*
- *For every $1 \leq k \leq n-1$, $H^k(G, \pi) = 0$.*

If $\mathbb{E}$ is a (commutative or non-commutative) L^p space with $1 < p < \infty$, then condition $\cos(\angle(P_\sigma, P_{\sigma'})) \leq \varepsilon$ can be verified if λ_τ is small enough for every $\tau \subseteq \triangle$, $\tau \in X(n-2)$ ("how small" is a function of p).

Thank you for listening